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On the Completeness of a Set of Translates

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We would like in this note to call attention to a somewhat surprising phenomenon concerning the completeness of a set of translates of a given function. With I an interval, f a continuous function defined on a slightly larger interval, and $\{t_k\}$ an infinite sequence of distinct numbers tending to zero, we consider the restriction to I of the translates of f by the amounts $\{t_k\}$, viewing them as members of $C(I)$, the space of all continuous functions on I. In general, of course, this yields too special a collection of functions for completeness in $C(I)$. Nevertheless, we will show that for f in a certain class, these translates are complete.

THEOREM. *Let I denote the finite interval $|x| \leqslant A$, let $B > A$, and suppose $\{t_k\}$ to be an infinite sequence of distinct points in the interval $|x| \leqslant B - A$, approaching* 0. *If $f(x) \not\equiv 0$ is the restriction to $|x| \leqslant B$ of a function $f^*(x)$, analytic and integrable over the x-axis, then the translates $\{f(t_k - x)\}$ are complete in $C(I)$.*

Proof. By the Hahn–Banach and Riesz Representation Theorems, a collection of functions is complete in $C(I)$ if and only if the only bounded measure $\mu(x)$ which annihilates them all is $\mu \equiv 0$. We will prove the theorem in this form. Accordingly, suppose $\mu(x)$ to be a bounded measure satisfying

$$\int_I f(t_k - x)\, d\mu(x) = 0, \qquad k = 1, 2,\ldots . \tag{1}$$

Let us, for definiteness, set $f(x) \equiv 0$ in $|x| > B$, and consider the continuous function $g(t) = \int_I f(t - x)\, d\mu(x)$. We observe that the values of $g(t)$ in $|t| \leqslant B - A$ depend on those of f in the interval $|x| \leqslant B$ only. Hence, on introducing

$$g^*(t) = \int_I f^*(t - x)\, d\mu(x), \tag{2}$$

we find that in the interval $|t| \leqslant B - A$ the functions g and g^* coincide, so that, by (1), $g^*(t)$ vanishes at the points $\{t_k\}$. But since f^* is analytic on the axis, so is g^*, and its vanishing on a real bounded infinite set implies its vanishing identically. Taking the Fourier transform of (2), and denoting by F^* and M the transforms of f^* and μ respectively, we conclude that $F^*(u)\,M(u) \equiv 0$. Since μ has a compact support, $M(u)$, being the restriction to the u-axis of an entire function in the $u + iv$ plane, has only isolated zeros; also $F^*(u) \not\equiv 0$, is bounded and continuous. Consequently, F^*M vanishes identically only if M does so, whereupon $\mu \equiv 0$. This concludes the proof of the Theorem. Of course, completeness of these translates in $L^p(I)$, $1 \leqslant p < \infty$, follows analogously. We remark that the example $f(x) \equiv 1$ for $|x| \leqslant B$ shows that the integrability condition imposed on f^* is not entirely superfluous. At the same time, there exist functions which approximate 1 arbitrarily closely on $|x| \leqslant B$ and have an integrable analytic extension. The phenomenon described by the theorem is therefore not stable.

If $\{\lambda_k\}$ is a regularly distributed set of points of unit density, and S is a single interval whose measure exceeds 2π, we cannot expect the exponentials $\{e^{i\lambda_k x}\}$ to be complete in $C(S)$. Completeness is possible, however, when S is the union of several intervals [2]. An application of the preceding theorem allows us to construct another example of this behavior:

COROLLARY. *Let $f(x) \not\equiv 0$ coincide in $|x| \leqslant \pi$ with a function analytic and integrable on the x-axis; set $f \equiv 0$ in $|x| > \pi$, and let $F(u)$ be the Fourier transform of $f(x)$. Extending F to an entire function in the $w = u + iv$ plane, let $\{\lambda_k\}$ denote the set of all zeros of F, and n_k the order of the zero λ_k. Given $\epsilon > 0$, let S, of total measure $4\pi - 4\epsilon$, consist of the two intervals $|x \pm \pi| \leqslant \pi - \epsilon$. Then the collection of functions*

$$E = \{x^m e^{i\lambda_k x},\ 0 \leqslant m \leqslant n_k - 1\}$$

is complete in $C(S)$.

Outline of Proof. As in the proof of the preceding result, suppose that $\mu(x)$ is a bounded measure supported on S which annihilates all of the functions in E; our goal is to show that $\mu \equiv 0$. To this end, we let $M(u)$ be the Fourier transform of μ,

$$M(u) = \frac{1}{\sqrt{2\pi}} \int_S e^{iux}\, d\mu(x), \tag{3}$$

and extend M to an entire function of exponential type in the w-plane. The annihilation condition ensures that the zeros of M include those of F, counting multiplicity, so that M/F is entire. We now choose a sufficiently smooth function $h(x) \not\equiv 0$, supported in $|x| \leqslant \epsilon/2$, whose Fourier transform $H(u)$ decreases so rapidly that $H(M/F)$ is square-integrable on the real line and of exponential type. The possibility of such a choice follows from the general results of [1], though, in the present simple case, it can be proved directly. By the Paley–Wiener theorem, the function $G \equiv H(M/F)$ is the Fourier transform of some $g(x)$, square-integrable and of compact support. Applying the inverse Fourier transform to the identity $GF \equiv HM$ yields

$$\int_{-\infty}^{\infty} g(x) f(t-x)\, dx \equiv \int_S h(t-x)\, d\mu(x). \tag{4}$$

Since f is assumed to be analytic on $|x| \leqslant \pi$, its support can be contained in no proper subinterval, and, by the choice of h, the convolution on the right-hand side of (4) vanishes for $|t| > 2\pi - (\epsilon/2)$; thus the Titchmarsh convolution theorem shows that the compact support of g lies in the interval $|x| \leqslant \pi - (\epsilon/2)$. Referring again to the right-hand side of (4), we note that it also vanishes for $|t| < \epsilon/2$, so that $\int_{|x| \leqslant \pi-(\epsilon/2)} g(x) f(t-x)\, dx = 0$ for t in a neighborhood of the origin. Now the preceding completeness theorem implies that $g(x) \equiv 0$, whence M and so also μ vanish identically. The Corollary is established. We emphasize that, since F is of exponential type π, the density of $\{\lambda_k\}$ cannot exceed 1.

In considering the zeros of an entire function F of exponential type, it is, broadly speaking, only their density that is delimited by the type; wide latitude is available in choosing or perturbing their position. When F is integrable on the real line, the type may be related to the diameter of the supporting set S of the Fourier transform of F. In this context, the Corollary may be interpreted as showing that requiring S to omit an interval, though not affecting the allowed growth of the function F, nevertheless exerts an influence, precise yet mysterious, on the location of its zeros.

References

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